A nonlocal connection between certain linear and nonlinear ordinary differential equations/oscillators

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V K Chandrasekar[1]**, M Senthilvelan**[1]**, Anjan Kundu**[2] **and M Lakshmanan**[1]

[1] Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University,
Tiruchirappalli 620 024, India
[2] Saha Institute of Nuclear Physics, Sector 1, Block AF, Bidhan Nagar, Kolkata 700 064, India

Abstract
We explore a nonlocal connection between certain linear and nonlinear ordinary differential equations (ODEs), representing physically important oscillator systems, and identify a class of integrable nonlinear ODEs of any order. We also devise a method to derive explicit general solutions of the nonlinear ODEs. Interestingly, many well-known integrable models can be accommodated into our scheme and our procedure thereby provides further understanding of these models.

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1. Introduction

In a recent paper, we have shown that the modified Emden-type equation with additional linear forcing,

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + \lambda x = 0, \tag{1}$$

where overdot denotes differentiation with respect to t and k and λ are arbitrary parameters, exhibits certain unusual nonlinear dynamical properties [1]. For a particular sign of the control parameter, namely, $\lambda > 0$, the frequency of oscillations of the nonlinear oscillator (1) is completely independent of amplitude and remains the same as that of the linear harmonic oscillator, thereby showing that the amplitude dependence of frequency is not necessarily a fundamental property of nonlinear dynamical phenomena [1]. In this case, namely, $\lambda > 0$, the system admits the explicit sinusoidal periodic solution

$$x(t) = \frac{A\sin(\omega t + \delta)}{1 - \left(\frac{k}{\omega}\right)A\cos(\omega t + \delta)}, \qquad 0 \leqslant A < \frac{\omega}{k}, \qquad \omega = \sqrt{\lambda}, \tag{2}$$

where A and δ are the arbitrary constants. The system (1) also exhibits certain other unusual properties. For more details on the dynamics of this equation one may refer to [1].

In the same paper, we have also noted that solution (2) can be derived unambiguously by introducing a nonlocal transformation,

$$U = x \, e^{k \int_0^t x(t') \, dt'}, \tag{3}$$

and transforming equation (1) to the linear harmonic oscillator equation,

$$\ddot{U} + \lambda U = 0. \tag{4}$$

For $\lambda > 0$, the solution is $U = A \sin(\omega t + \delta)$, where A and δ are the arbitrary constants and the frequency, $\omega = \sqrt{\lambda}$, is independent of the amplitude. Using (3) one finds that

$$\dot{x} = \frac{\dot{U}}{U} x - kx^2. \tag{5}$$

Integrating Riccati-type equation (5) and absorbing the integration constant with the existing ones, one can obtain solution (2). We mention here that solution (2) can be derived in a number of ways, for more details one may again refer to [1, 2]. One may also note that in the case $\lambda = 0$, transformation (3) modifies equation (1) to the free particle equation. The linearization and integrability properties of equation (1) with $\lambda = 0$ have received a considerable attention in the nonlinear dynamics literature, see for example, [3–6].

Now the question that naturally arises is whether the nonlocal transformation which connects the linear and nonlinear oscillators, namely equations (1) and (4), is a rare one or there exists a wider class of nonlinear dynamical systems that are connected with linear oscillators in a hidden way? Our studies reveal that there exists a class of linear oscillators that are connected with nonlinear oscillator equations through nonlocal transformations. For example, let us consider a damped linear harmonic oscillator, instead of the undamped linear harmonic oscillator, that is,

$$\ddot{U} + c_1 \dot{U} + c_2 U = 0, \tag{6}$$

where c_1 and c_2 are arbitrary parameters, and consider a more general form of nonlocal transformation, that is,

$$U = x(t) \, e^{\int_0^t f(x(t')) \, dt'}, \tag{7}$$

where $f(x(t))$ is an arbitrary function of x. Now substituting (7) into (6) we get a nonlinear ODE of the form

$$\ddot{x} + (2f + xf' + c_1)\dot{x} + (f^2 + c_1 f + c_2)x = 0, \tag{8}$$

where the prime denotes the differentiation with respect to x. Interestingly, one can now see that for certain specific forms of the function f, the associated nonlinear ODEs become the well-known and well-studied models in the current nonlinear dynamics literature. To cite a few, first let us consider the case $f(x) = kx$ in (7). The associated nonlinear ODE,

$$\ddot{x} + (c_1 + 3kx)\dot{x} + k^2 x^3 + c_1 k x^2 + c_2 x = 0, \tag{9}$$

is nothing but the generalized modified Emden-type equation (MEE) whose integrability properties have been studied in detail in [3–10]. It is one of the linearizable second-order nonlinear ODEs which admits eight parameter Lie point symmetries [4].

Choosing $f(x) = kx^2$ in (7), one gets the generalized force-free Duffing–van der Pol oscillator (DVP) equation

$$\ddot{x} + (c_1 + 4kx^2)\dot{x} + k^2 x^5 + kc_1 x^3 + c_2 x = 0, \tag{10}$$

as its nonlinear counterpart. More details on the mathematical aspects and the underlying dynamics of this equation can be found in [9–13]. On the other hand, fixing $f(x) = \frac{k}{x^2}$, one gets another integrable nonlinear second-order ODE of the form [9]

$$\ddot{x} + c_1 \dot{x} + c_2 x + \frac{kc_1}{x} + \frac{k^2}{x^3} = 0. \tag{11}$$

The examples illustrated above clearly demonstrate that one can systematically classify a class of integrable nonlinear ODEs through this way.

Motivated by these observations, in this paper, we carry out a detailed investigation on the nonlocal connection between certain linear ODEs and their nonlinear counterparts of any order. To make our study a systematic one we start our investigations at the level of second order and end with the nth-order equations. Besides constructing integrable nonlinear equations we also point out the importance of these nonlinear equations and obtain their solutions and thereby bring out the significance of these systems.

The plan of the paper is as follows. In section 2, we make a detailed study on the nonlocal connection between damped linear harmonic oscillator equation and certain nonlinear oscillator equations of second order and describe a method of deriving general solutions of the nonlinear ODEs. We extend this procedure to third- and nth-order ODEs in sections 3 and 4, respectively. We present a general theory which connects the variable coefficient linear ODEs with nonlinear ODEs and the method of finding the general solution in section 5. We present our conclusion in section 6.

2. Second-order ODEs

To begin with let us consider a linear second-order ODE of the form (6), whose general solution is $U = a(t)$, where $a(t)$ is a known function. Now we consider a nonlocal transformation of the form

$$U = x^n \, e^{\int_0^t (\beta(t')x^m + \gamma(t'))dt'}, \tag{12}$$

where n and m are the constants and $\beta(t)$ and $\gamma(t)$ are the arbitrary functions of t and substitute it into (6) so that the latter becomes a nonlinear second-order ODE of the form

$$\ddot{x} + (n-1)\frac{\dot{x}^2}{x} + \frac{\beta^2}{n}x^{2m+1} + b_1(t,x)\dot{x} + b_2(t)x^{m+1} + b_3(t)x = 0, \tag{13}$$

where

$$\begin{aligned}
b_1(t,x) &= \frac{1}{n}(2n\gamma + nc_1 + (m+2n)\beta x^m), \\
b_2(t) &= \frac{1}{n}(\dot{\beta} + 2\gamma\beta + c_1\beta), \\
b_3(t) &= \frac{1}{n}(\dot{\gamma} + \gamma^2 + \gamma c_1 + c_2).
\end{aligned} \tag{14}$$

Obviously transformation (12) corresponds to the special case, $f(x) = \beta(t)x^m$, $\gamma(t) = 0$, in transformation (7) but with the prefactor of the exponential function taken as x^n, where n is arbitrary. We have been motivated to choose this form due to the examples illustrated in the introduction. In all these cases, one might have noted that the function $f(x)$ is a simple polynomial function. As a consequence, we would like to generalize this form and see the outcome. Such a generalization is indicated in section 5. However, here we note that the form (13) itself yields several important nonlinear ODEs as see below.

Now the question is how to construct the general solution for the nonlinear equation (13) from the known solution of (6). This can be done by using the identity,

$$\frac{\dot{U}}{U} = \frac{n\dot{x}}{x} + \beta(t)x^m + \gamma(t), \tag{15}$$

derived from equation (12). Since $U = a(t)$, where $a(t)$ is a known function, the solution of the second-order linear ODE (6), equation (15) can be brought to the form

$$\dot{x} = (\hat{a}(t) - \gamma(t))\frac{x}{n} - \frac{\beta(t)}{n}x^{m+1}, \tag{16}$$

where $\hat{a} = \frac{\dot{a}}{a}$. Solving equation (16) we get the general solution for equation (13) in the form

$$x(t) = e^{\frac{1}{n} \int_0^t (\hat{a}(t') - \gamma(t'))dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t') e^{\frac{m}{n} \int_0^{t'} (\hat{a}(t'') - \gamma(t''))dt''} \right) dt' \right]^{\frac{-1}{m}}. \quad (17)$$

Note that equation (17) contains three arbitrary constants (two in $\hat{a}(t)$, since $a(t)$ is a general solution of the second-order equation (6) and another constant C). However, one can absorb one arbitrary constant with the other two and rewrite the solution in such a way that it contains only two arbitrary constants, as we see below.

Let us assume that the solution of (6) be written in the form

$$U(t) = a(t) = I_1 e^{m_1 t} + I_2 e^{m_2 t}, \quad (18)$$

where the I_i's, $i = 1, 2$, are the integration constants and the m_i's, $i = 1, 2$, are the roots of the auxiliary equation associated with the ODE (6). From (18) we get

$$\hat{a}(t) = \frac{\dot{a}}{a} = \frac{\dot{U}}{U} = \frac{m_1 \hat{I}_1 e^{m_1 t} + m_2 e^{m_2 t}}{\hat{I}_1 e^{m_1 t} + e^{m_2 t}}, \quad (19)$$

where $\hat{I}_1 = \frac{I_1}{I_2}$. Integrating both sides of equation (19) we get

$$\int_0^t \hat{a}(t') \, dt' = \log(\hat{I}_1 e^{m_1 t} + e^{m_2 t}) = \log\left(\frac{a(t)}{I_2} \right). \quad (20)$$

Substituting expression (20) into equation (17), we get a general solution for equation (13) in the form

$$x(t) = \left(\frac{a(t)}{I_2} \right)^{\frac{1}{n}} e^{\frac{-1}{n} \int_0^t \gamma(t')dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t') \left(\frac{a(t')}{I_2} \right)^{\frac{m}{n}} e^{\frac{-m}{n} \int_0^{t'} \gamma(t'')dt''} \right) dt' \right]^{\frac{-1}{m}}. \quad (21)$$

Now the general solution (21) contains only two arbitrary constants, namely, $\hat{I}_1$ and C. In the following, we consider certain important sub-cases of the linear oscillator (6) and discuss their nonlinear counterparts and their physical significance.

2.1. Case (i) $c_1 = 0$ and $c_2 = 0$

In this case the linear equation (6) is nothing but the free particle equation. The associated nonlinear equation can be fixed from (13) by restricting $c_1 = 0$ and $c_2 = 0$. The general solution of the nonlinear equation turns out to be

$$x(t) = (t + I_1)^{\frac{1}{n}} e^{\frac{-1}{n} \int_0^t \gamma(t')dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t')(t' + I_1)^{\frac{m}{n}} e^{-\frac{m}{n} \int_0^{t'} \gamma(t'')dt''} \right) dt' \right]^{\frac{-1}{m}}, \quad (22)$$

where I_1 and C are the arbitrary constants.

The interesting fact is that equation (13) contains a family of important nonlinear ODEs. To mention one such example let us choose $n = 1, \gamma(t) = 0$ and $\beta(t) = k$ so that equation (13) now becomes

$$\ddot{x} + (m + 2)kx^m \dot{x} + k^2 x^{2m+1} = 0. \quad (23)$$

The invariance, integrability properties and direct linearization through generalized transformation of equation (23) can be found in [6, 9, 10]. From our above results, the general solution of (23) can be fixed easily from (22) in the form

$$x(t) = (t + I_1) \left[C + \frac{mk}{(m + 1)} (I_1 + t)^{m+1} \right]^{\frac{-1}{m}}, \quad (24)$$

which exactly coincides with the results obtained through other methods [6, 9, 10].

2.2. Case (ii) $c_1 = 0$ and $c_2 = constant$

In this case, from equation (6), we have the linear harmonic oscillator equation at our hand. The nonlocal transformation (12) transforms the linear harmonic oscillator equation, $\ddot{U} + c_2 U = 0$, to the nonlinear ODE (13) with $c_1 = 0$. The general solution of (13) turns out to be

$$x(t) = [\cos(\omega t + \delta)]^{\frac{1}{n}} \, e^{\frac{-1}{n} \int_0^t \gamma(t')dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t') \, e^{-\frac{m}{n} \int_0^{t'} \gamma(t'')dt''} \cos^{\frac{m}{n}}(\omega t' + \delta) \right) dt' \right]^{\frac{-1}{m}},$$
(25)

where $\omega = \sqrt{c_2}$ and δ and C are the arbitrary constants.

Interestingly, one can deduce certain important nonlinear ODEs from (13) with $c_1 = 0$. One such interesting case is $n = 1, m = 1, \gamma(t) = 0$ and $\beta(t) = k$. In this case we get equation (1). The unusual dynamical properties exhibited by this equation has already been pointed out in the introduction.

Now we choose $n = 1, m = -2, \gamma(t) = 0$ and $\beta(t) = k$ in equation (13) so that the latter becomes

$$\ddot{x} + c_2 x + \frac{k^2}{x^3} = 0,$$
(26)

which is another important nonlinear ODE which arises in different areas of physics and has been studied in detail in [14, 15]. The solution can be derived from (25) in the form

$$x(t) = \cos(\omega t + \delta) \left[C - \frac{2k}{\omega} \tan(\omega t + \delta) \right]^{\frac{1}{2}}, \qquad \omega = \sqrt{c_2}.$$
(27)

Note that the solution defines a well-defined harmonic periodic oscillation with period $T = \frac{2\pi}{\sqrt{c_2}}$ which is the same as that of the unperturbed simple harmonic oscillator. This nonlinear oscillator again supports our argument that the amplitude dependence of frequency is not necessarily a fundamental property of nonlinear oscillations.

On the other hand, fixing $n = 1, m = 2, \gamma(t) = -k_2$ and $\beta(t) = k_1$ in equation (13) we get

$$\ddot{x} + (4k_1 x^2 - 2k_2)\dot{x} + k_1^2 x^5 - 2k_1 k_2 x^3 + (c_2 + k_2^2)x = 0,$$
(28)

which is nothing but the generalized force-free Duffing–van der Pol nonlinear oscillator equation [9, 11–13]. The general solution can again be fixed from (25) as

$$x(t) = \cos(\omega t + \delta) \left[C \, e^{-2k_2 t} + \frac{k_1}{2k_2} + \frac{k_1}{2(\omega^2 + k_2^2)}(k_2 \cos 2(\omega t + \delta) + \omega \sin 2(\omega t + \delta)) \right]^{-\frac{1}{2}},$$
(29)

which exactly coincides with the result obtained through other methods [9, 11–13].

Finally, let us consider the case $n = 1, m = q, \gamma(t) = -k_2$ and $\beta(t) = k_1$ in equation (13). In this case we end up with the following nonlinear ODE:

$$\ddot{x} + ((q + 2)k_1 x^q - 2k_2)\dot{x} + [(k_1 x^q - k_2)^2 + c_2]x = 0.$$
(30)

Equation (30) admits the following general solution:

$$x(t) = \cos(\omega t + \delta) \, e^{k_2 t} \left[C + qk_1 \int_0^t (e^{qk_2 t'} \cos^q(\omega t' + \delta)) \, dt' \right]^{-\frac{1}{q}}.$$
(31)

We mention here that for even values of q and positive values of k_1 and k_2 equation (30) exhibits limit cycle oscillations [12].

2.3. Case (iii) $c_1, c_2 = constant$

In this case the nonlocal transformation connects the damped harmonic oscillator ODE with the nonlinear ODE (13). From (21), the general solution of equation (13) with c_1 and c_2 taking constant values turns out to be

$$x(t) = \left(e^{(\frac{-c_1+\omega}{2})t} + I_1 e^{-(\frac{c_1+\omega}{2})t}\right)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'}$$
$$\times \left[C + \frac{m}{n}\int_0^t \left(\beta(t')\left(\frac{I_1 + e^{\omega t'}}{e^{(\frac{c_1+\omega}{2})t'}}\right)^{\frac{m}{n}} e^{-\frac{m}{n}\int_0^{t'} \gamma(t'')dt''}\right) dt'\right]^{\frac{-1}{m}}, \quad (32)$$

where $\omega = \sqrt{c_1^2 - 4c_2}$ and I_1 and C are the arbitrary constants.

In this case also one can identify several physically interesting nonlinear oscillator equations. For example, choosing $n = 1, m = 1, \gamma(t) = 0$ and $\beta(t) = k$ in equation (13) we get an equation of the form (9). The general solution can be written from (32) in the form

$$x(t) = \left(\frac{2c_2(I_1 + e^{\omega t})}{C e^{(\frac{c_1+\omega}{2})t} - k(c_1 - \omega)I_1 - k(c_1 + \omega) e^{\omega t}}\right), \quad (33)$$

where $\omega = \sqrt{c_1^2 - 4c_2}$. Indeed our result exactly coincides with the result obtained in the [8, 9].

The choice $n = 1, m = 2, \gamma(t) = 0$ and $\beta(t) = k$ leads to an equation of the form (10). The general solution of (10) can be fixed easily from (32) in the form

$$x(t) = \left(\frac{2c_1c_2(I_1 + e^{\omega t})^2}{C e^{(\frac{c_1+\omega}{2})t} - kI_1^2c_1(c_1 - \omega) - kc_1(c_1 + \omega) e^{2\omega t} - 8kI_1c_2 e^{\omega t}}\right)^{\frac{1}{2}}, \quad (34)$$

which indeed the general solution of the genaralized force-free DVP oscillator equation [9].

In this section, besides the general case, we considered three different linear equations, namely, free particle equation, linear harmonic oscillator and damped linear harmonic equations and demonstrated how they are related to the well-known nonlinear models through nonlocal transformations. In the following section, we extend the theory to third-order ODEs and study the outcome.

3. Third-order ODEs

Let us consider a linear third-order ODE of the form

$$\dddot{U} + c_1\ddot{U} + c_2\dot{U} + c_3U = 0, \quad (35)$$

where c_1, c_2 and c_3 are the arbitrary constants. The nonlocal transformation (12) transforms (35) to the nonlinear ODE of the form

$$\dddot{x} + \left[3(n-1)\frac{\dot{x}}{x} + d_1(t,x)\right]\ddot{x} + (n-1)(n-2)\frac{\dot{x}^3}{x^2} + d_2(t,x)\dot{x}^2$$
$$+ d_3(t,x)\dot{x} + \frac{\beta^3}{n}x^{3m+1} + d_4(t)x^{2m+1} + d_5(t)x^{m+1} + d_6(t)x = 0, \quad (36)$$

where

$$d_1(t, x) = \frac{1}{n}(3n\gamma + nc_1 + \beta(m + 3n)x^m),$$

$$d_2(t, x) = \frac{1}{n}((m(m + 2n) + (n - 1)(m + 3n))\beta x^{m-1} + n(n - 1)(3\gamma + c_3)x^{-1}),$$

$$d_3(t, x) = \frac{1}{n}((3\gamma\beta(m + 2n) + \dot{\beta}(2m + 3n) + c_1(m + 2n)\beta)x^m + 3\beta^2(m + n)x^{2m}$$
$$+ n(3\dot{\gamma} + 3\gamma^2 + 2c_1\gamma + c_2)), \tag{37}$$

$$d_4(t) = \frac{1}{n}(3\beta\dot{\beta} + 3\gamma\beta^2 + \beta^2 c_1),$$

$$d_5(t) = \frac{1}{n}(\ddot{\beta} + 3(\dot{\gamma}\beta + \gamma\dot{\beta}) + 3\beta\gamma^2 + (c_1\dot{\beta} + c_2\beta) + 2c_1\gamma\beta),$$

$$d_6(t) = \frac{1}{n}(\ddot{\gamma} + 3\gamma\dot{\gamma} + \gamma^3 + c_1(\dot{\gamma} + \gamma^2) + \gamma c_2 + c_3).$$

To obtain the exact solution, $x(t)$, of equation (36) we again apply the results of equations (15)–(17) to the present case. Since we are using the same form of nonlocal transformation (12) to transform the nonlinear ODE (36) to the linear ODE (35), the general solution of (36) looks exactly the same form as (17). The only difference is that the general solution, in the present case, contains four integration constants (three of them come out from the solution of the linear ODE (35) and the remaining one from solving (16)). Now let us rewrite the general solution in such a way that it contains only three arbitrary constants by absorbing the fourth one. To do so let us follow the same procedure which we adopted in the case of second-order ODEs.

Let us write the general solution of equation (35) in the form

$$U(t) = a(t) = I_1 e^{m_1 t} + I_2 e^{m_2 t} + I_3 e^{m_3 t}, \tag{38}$$

where the I_i's, $i = 1, 2, 3$, are the integration constants and the m_i's, $i = 1, 2, 3$, are the roots of the auxiliary equation associated with the differential equation (35). From (38) we get

$$\hat{a}(t) = \frac{\dot{a}}{a} = \frac{\dot{U}}{U} = \frac{m_1\hat{I}_1 e^{m_1 t} + m_2\hat{I}_2 e^{m_2 t} + m_3 e^{m_3 t}}{\hat{I}_1 e^{m_1 t} + \hat{I}_2 e^{m_2 t} + e^{m_3 t}}, \tag{39}$$

where $\hat{I}_i = \frac{I_i}{I_3}$, $i = 1, 2$. Integrating both sides of equation (39) we get

$$\int_0^t \hat{a}(t')\,dt' = \log(\hat{I}_1 e^{m_1 t} + \hat{I}_2 e^{m_2 t} + e^{m_3 t}) = \log\left(\frac{a(t)}{I_3}\right). \tag{40}$$

Substituting expression (40) into (17), we get the general solution for equation (36) in the form

$$x(t) = \left(\frac{a(t)}{I_3}\right)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'}\left[C + \frac{m}{n}\int_0^t \left(\beta(t')\left(\frac{a(t')}{I_3}\right)^{\frac{m}{n}} e^{\frac{-m}{n}\int_0^{t'} \gamma(t'')dt''}\right)dt'\right]^{\frac{-1}{m}}. \tag{41}$$

We note that the general solution (41) now contains only three arbitrary constants, namely, $\hat{I}_1, \hat{I}_2$ and C.

Since we have three arbitrary parameters in the linear ODE (35), one can consider certain specific sub-cases of physical interest. In the following, we discuss some of them.

3.1. Case (i) $c_i = 0, i = 1, 2, 3$

In this case, from equation (35), we have a simple linear equation $\frac{d^3U}{dt^3} = 0$ which can be connected to the third-order nonlinear ODE (36) (with $c_1, c_2, c_3 = 0$) through the nonlocal

transformation (12). The general solution of (36) can be fixed easily from (41) in the form

$$x(t) = \left(\frac{1}{2}t^2 + I_1 t + I_2\right)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'}$$

$$\times \left[C + \frac{m}{n}\int_0^t \left(\beta(t')\left(\frac{1}{2}t'^2 + I_1 t' + I_2\right)^{\frac{m}{n}} e^{-\frac{m}{n}\int_0^{t'}\gamma(t'')dt''}\right)dt'\right]^{\frac{-1}{m}}, \tag{42}$$

where I_1, I_2 and C are the arbitrary constants.

Equation (36) includes a class of important nonlinear systems. For example, choosing $n = 1, m = 1, \gamma(t) = 0$ and $\beta(t) = k$ equation (36) becomes

$$\dddot{x} + 4kx\ddot{x} + 3k\dot{x}^2 + 6k^2x^2\dot{x} + k^3x^4 = 0. \tag{43}$$

Equation (43) is a special case of the Chazy equation XII (with $N = 2$ and parametric restrictions $A = 0, B = 0$ in [18]), which has been studied in detail in [16–22]. The general solution can be fixed easily from (42) in the form

$$x(t) = \left(\frac{\frac{kt^2}{2} + I_1 t + I_1 I_2}{\frac{k^2t^3}{6} + \frac{kI_1}{2}t^2 + kI_1 I_2 t + I_1 I_3}\right). \tag{44}$$

3.2. Case (ii) $c_3 = 0$, $c_2 = 0$ and $c_1 = constant$

In this case, the linear equation (35) becomes $\ddot{U} + c_1\dot{U} = 0$ and the nonlocal transformation (12) transforms this ODE to the form (36) with $c_2, c_3 = 0$. In this case we have the general solution of the form

$$x(t) = (e^{-c_1 t} + I_1 t + I_2)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'}$$

$$\times \left[C + \frac{m}{n}\int_0^t \left(\beta(t')(e^{-c_1 t'} + I_1 t' + I_2)^{\frac{m}{n}} e^{-\frac{m}{n}\int_0^{t'}\gamma(t'')dt''}\right)dt'\right]^{\frac{-1}{m}}, \tag{45}$$

where I_1, I_2 and C are the arbitrary constants. Choosing $n = 1, m = 1, \gamma(t) = 0$ and $\beta(t) = k$ equation (36) becomes

$$\dddot{x} + (c_1 + 4kx)\ddot{x} + 3k\dot{x}^2 + 3k(c_1 + 2kx)x\dot{x} + (c_1 + kx)k^2x^3 = 0. \tag{46}$$

The general solution of equation (46) can be fixed easily from (45) in the form

$$x(t) = \left(\frac{e^{-c_1 t} + I_1 t + I_2}{C + \frac{k}{2c_1}(2e^{-c_1 t} + c_1 I_1 t^2 + 2c_1 I_2 t)}\right). \tag{47}$$

3.3. Case (iii) $c_1 = 0$, $c_2 = 0$ and $c_3 = constant$

In this case, the linear ODE (35) assumes the form $\ddot{U} + c_3 U = 0$ and the nonlocal transformation (12) transforms this equation, $\ddot{U} + c_3 U = 0$, to the nonlinear ODE (36) with $c_1, c_2 = 0$. The general solution of (36) turns out to be

$$x(t) = \left(I_1 e^{-kt} + e^{\frac{k}{2}t}\cos\left(\frac{\sqrt{3}k}{2}t + \delta\right)\right)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'}\left[C + \frac{m}{n}\int_0^t \left(\beta(t') e^{-\frac{m}{n}\int_0^{t'}\gamma(t'')dt''}\right.\right.$$

$$\left.\left.\times \left(I_1 e^{-kt'} + e^{\frac{k}{2}t'}\cos\left(\frac{\sqrt{3}k}{2}t' + \delta\right)\right)^{\frac{m}{n}}\right)dt'\right]^{\frac{-1}{m}}, \tag{48}$$

where $c_3 = k^3$ and I_1, C and δ are the arbitrary constants. By choosing $n = 1$, $m = 1$, $\gamma(t) = 0$ and $\beta(t) = k$ in equation (36) one can get a special case of the Chazy equation XII (with $N = 2$ and parametric restrictions $A = c_3$ and $B = 0$ in [18])

$$\dddot{x} + 4kx\ddot{x} + 3k\dot{x}^2 + 6k^2x^2\dot{x} + k^3x^4 + c_3x = 0. \tag{49}$$

The general solution of (49) can be fixed easily from (48) in the form

$$x(t) = \left(\frac{I_1\,\mathrm{e}^{-kt} + \mathrm{e}^{\frac{k}{2}t}\cos\left(\frac{\sqrt{3}k}{2}t + \delta\right)}{C - \frac{k}{2k}\left(2I_1\,\mathrm{e}^{-kt} - \mathrm{e}^{\frac{k}{2}t}\left(\cos\left(\frac{\sqrt{3}k}{2}t + \delta\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}k}{2}t + \delta\right)\right)\right)} \right). \tag{50}$$

3.4. Case (iv) $c_i = constant$, $i = 1, 2, 3$

In this case the nonlocal transformation transforms the nonlinear ODE (36) to the linear ODE (35). The general solution of the latter becomes

$$x(t) = (\mathrm{e}^{m_1t} + I_1\,\mathrm{e}^{m_2t} + I_2\,\mathrm{e}^{m_3t})^{\frac{1}{n}}\,\mathrm{e}^{\frac{-1}{n}\int_0^t \gamma(t')\mathrm{d}t'}$$

$$\times \left[C + \frac{m}{n}\int_0^t (\beta(t')(\mathrm{e}^{m_1t'} + I_1\,\mathrm{e}^{m_2t'} + I_2\,\mathrm{e}^{m_3t'})^{\frac{m}{n}}\,\mathrm{e}^{-\frac{m}{n}\int_0^{t'}\gamma(t'')\mathrm{d}t''})\,\mathrm{d}t' \right]^{\frac{-1}{m}}, \tag{51}$$

where I_1, I_2 and C are the arbitrary constants and $m_i^3 + c_1m_i^2 + c_2m_i + c_3 = 0$, $i = 1, 2, 3$, and $m_{2,3} = \frac{1}{2}\left[-(m_1 + c_1) \pm \sqrt{-3m_1^2 - 2c_1m_1 + c_1^2 - 4c_2} \right]$.

Equation (36) includes a class of important nonlinear systems. For $n = 1$, $m = 1$, $\gamma(t) = 0$ and $\beta(t) = k$ equation (36) becomes

$$\dddot{x} + (c_1 + 4kx)\ddot{x} + 3k\dot{x}^2 + (c_2 + 3kc_1 + 6k^2x)x\dot{x} + (c_1 + kx)k^2x^3 + c_2kx^2 + c_3x = 0. \tag{52}$$

Equation (52) is a generalized version of the above special cases (43) and (49) of the Chazy equation XII, with additional terms. The general solution of (52) can be fixed easily from (51) in the form

$$x(t) = (\mathrm{e}^{m_1t} + I_1\,\mathrm{e}^{m_2t} + I_2\,\mathrm{e}^{m_3t})\left[C + k\left(\frac{\mathrm{e}^{m_1t}}{m_1} + \frac{I_1}{m_2}\mathrm{e}^{m_2t} + \frac{I_2}{m_3}\mathrm{e}^{m_3t} \right) \right]^{-1}. \tag{53}$$

To our knowledge, the above solution is a new one. For a detailed description of Chazy class of third-order ordinary differential equations, their analytic properties and their general solutions one may refer to [18], besides the original work of Chazy [16].

In this section, we demonstrated the nonlocal connection that exists between linear and nonlinear ODEs of third order. In the following section, we extend the theory to nth-order ODEs.

4. nth-order equations

Let us consider a most general linear ODE of the form

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} + c_1\frac{\mathrm{d}^{(n-1)}}{\mathrm{d}t^{(n-1)}} + \cdots + c_{n-1}\frac{\mathrm{d}}{\mathrm{d}t} + c_n \right)U(t) = 0, \tag{54}$$

where the c_i's, $i = 1, 2, \ldots, n$, are the arbitrary constants. The nonlocal transformation (12) connects (54) to the nonlinear ODE of the form

$$\left(D_h^{(n)} + c_1D_h^{(n-1)} + \cdots + c_{n-1}D_h^{(1)} + c_n \right)x^n = 0, \tag{55}$$

where $D_h^{(n)} = \left(\frac{d}{dt} + \beta(t)x^m + \gamma(t)\right)^n$. For $n = 2$ and 3 equation, (55) coincides with (13) and (36) respectively. Repeating the procedure described in sections 2 and 3 one can derive the general solution of the nth-order nonlinear ODE, (55), in the form

$$x(t) = e^{\frac{1}{n}\int_0^t (\hat{a}(t') - \gamma(t'))dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t') e^{\frac{m}{n}\int_0^{t'} (\hat{a}(t'') - \gamma(t''))dt''} \right) dt' \right]^{\frac{-1}{m}}. \quad (56)$$

Here again $\hat{a}(t) = \frac{\dot{a}}{a}$, where a is the known solution of equation (54). The solution contains $n + 1$ integration constants, namely, $I_1, I_2, \ldots, I_n$ and C. However, as was done in the second- and third-order ODEs, respectively, we can absorb one integration constant and rewrite the solution in such a way that it contains only n integration constants. Doing so we arrive at

$$x(t) = \left(\frac{a(t)}{I_n}\right)^{\frac{1}{n}} e^{\frac{-1}{n}\int_0^t \gamma(t')dt'} \left[C + \frac{m}{n} \int_0^t \left(\beta(t') \left(\frac{a(t')}{I_n}\right)^{\frac{m}{n}} e^{\frac{-m}{n}\int_0^{t'} \gamma(t'')dt''} \right) dt' \right]^{\frac{-1}{m}}. \quad (57)$$

By appropriately fixing the constants and arbitrary functions in equation (55) one can deduce the nonlinear equation of interest. The general solution can also be fixed unambiguously from equation (57).

5. General theory

So far we focused our attention only on the cases in which the nonlocal transformation that connects the nonlinear ODEs with constant coefficient linear ODEs. However, one can also explore the hidden connection between nonlinear ODEs and variable coefficient linear ODEs. In the following we briefly discuss the underlying theory.

Let us begin with an nth-order linear ODE of the form

$$\left(\frac{d^n}{dt^n} + c_1(t)\frac{d^{(n-1)}}{dt^{(n-1)}} + \cdots + c_{n-1}(t)\frac{d}{dt} + c_n(t) \right) U(t) = 0, \quad (58)$$

where the c_i's, $i = 1, 2, \ldots, n$, are the functions of t, whose general solution be $U = a(t)$, where $a(t)$ is some known function. Now we consider a general nonlocal transformation of the form

$$U(t) = g(t, x) e^{\int_0^t f(t', x)dt'}, \quad (59)$$

where $f(t, x)$ and $g(t, x)$ are the arbitrary functions of t and x and substitute it into (58) so that the latter becomes a nonlinear ODE of the form

$$\left(D_h^{(n)} + c_1(t)D_h^{(n-1)} + \cdots + c_{n-1}(t)D_h^{(1)} + c_n(t) \right) g(t, x) = 0, \quad (60)$$

where $D_h^{(n)} = \left(\frac{d}{dt} + f(t, x)\right)^n$. Since we have assumed that the multiplicative function $g(t, x)$ is a function of the variables t and x, the associated nonlinear ODE, (60), takes a very complicated form. However, the task is to deduce the interesting cases from (60) which can be integrated explicitly. Again using the identity

$$\frac{\dot{U}}{U} = \frac{g_x \dot{x} + g_t}{g} + f, \quad (61)$$

with $U = a(t)$, equation (61) can be brought to the form

$$\dot{x} = (\hat{a}(t) - f(t, x))\frac{g(t, x)}{g_x(t, x)} - \frac{g_t(t, x)}{g_x(t, x)}, \qquad \hat{a} = \frac{\dot{a}}{a}. \quad (62)$$

Now, it is well known that only for certain specific forms of f and g, equation (62) can be integrated [23]. At least for these cases, the general solution for the nonlinear ODE (60) can

be obtained explicitly. In this way one can classify a class of integrable nonlinear ODEs of any order through nonlocal transformations of the form (59).

We note that equation (60) can also be obtained from equation (58) by using a simple 'gauge transformation'

$$\frac{d^n}{dt^n} \to D_h^{(n)} = \left(\frac{d}{dt} + f(t, x) \right)^n, \qquad f(t, x) = \frac{d}{dt} h(t, x)$$

and

$$U(t) \to g(x, t).$$

Finally, we mention that in the case $f(t, x) = 0$ the nonlocal transformation, (59), becomes purely a local one and the resultant linearizable equations can be directly obtained from equation (60).

6. Conclusion

In this paper, we have developed a novel way of identifying integrable nonlinear ODEs by connecting linear and nonlinear oscillator equations of any order through suitable nonlocal transformations. The proposed method is simple and straightforward. More importantly, our procedure not only identifies a class of integrable nonlinear ODEs of any order but also unambiguously gives their underlying solutions and thereby leads to the complete understanding of the dynamics of the given nonlinear system. As we have pointed out in the introduction, the modified Emden-type equation (1) admits certain unusual nonlinear dynamical properties. The dynamical properties underlying other integrable nonlinear ODEs still remain to be explored in detail and are possibly worth investigating further. It is also of interest to investigate the existence of the above types of nonlocal connections in the case of partial differential equations as well, which we hope to pursue further.

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